①

(A),(B) Not complete

Let A be any non-measurable subset of $\mathbb{R}$.

Then, $\{0\}\times A$ is of measure zero & is lebesgue measurable.

But $\{0\}\times A \notin \mathcal{B}(\mathbb{R}^2)$ & $\{0\}\times A \notin \mathcal{L}(\mathbb{R})\times\mathcal{L}(\mathbb{R})$.

(C) complete.

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② (B) True.

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$$\int_0^b |g(x)|\,dx \le \int_0^b\int_x^b \left|\frac{f(t)}{t}\right| dt\,dx \qquad \left(\because \frac{|f(t)|}{t} \geq 0 \text{ on } [0,b]\times[0,b]\right)$$

$$= \int_0^b\int_0^t \frac{|f(t)|}{t}\,dx\,dt$$

$$= \int_0^b |f(t)|\,dt < \infty.$$

$\Rightarrow g$ is integrable &

$$\int_0^b g(x)\,dx = \int_0^b\int_x^b \frac{f(t)}{t}\,dt\,dx$$

$$= \int_0^b\int_0^t \frac{f(t)}{t}\,dx\,dt \qquad (\because \text{by Fubini's theorem})$$

$$= \int_0^b f(t)\,dt$$

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③ (A) if $a>-n$.

$$\int_{\mathbb{R}^n} f(x)\,dx = \int_0^\infty \int_{\partial B(0,r)} f(x)\,d\sigma\,dr$$

$$= \int_0^1 \int_{\partial B(0,r)} r^a\,d\sigma\,dr$$

$$= \int_0^1 c\cdot r^a\cdot r^{n-1}\,dr$$

$$= c\int_0^1 r^{n+a-1}\,dr$$

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$$\int_0^1 r^{a+n-1}\,dr < \infty \quad \text{if} \quad n+a-1 > -1 \text{ i.e.) } a > -n$$

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④ B) $\int_{\mathbb{R}^n} f(x)\,dx = c\int_1^\infty r^{a+n-1}\,dr < \infty$ if $n+a-1 < -1$ i.e. $a < -n$.

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⑤ (A) if $n=2$ take $a=-1$

$$\therefore \int_{\mathbb{R}^n} f(x)\,dx = \int_0^\infty \int_{\partial B(0,r)} f(x)\,d\sigma\,dr$$

$$= \int_0^1 \int_{\partial B(0,r)} \frac{1}{(1-r^2)}\,d\sigma\,dr$$

$$= c\int_0^1 \frac{r}{1-r^2}\,dr$$

$$= \infty$$

(B) if $a>0$, $f(x) \le 1$ on $\mathbb{R}^n$.

$$\Rightarrow \int_{\mathbb{R}^n} f(x)\,dx = \int_{|x|\le 1} f(x)\,dx \le \int_{|x|\le 1} dx < \infty.$$

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⑥ (A) True.

if $a<b$, then $\{x : |f(x)|>b\} \subseteq \{x : |f(x)|>a\}$

$\Rightarrow d_f(b) \le d_f(a)$

& the continuity from the right follows from the fact that if $E_n \subseteq E_{n+1}$, $\forall n$ & $E = \bigcup_n E_n$, then $m(E) = \lim_{n\to\infty} E_n$.

(B) $\lambda d_f(\lambda) \le \int_{E_\lambda} |f|\,dm$, where $E_\lambda = \{x : |f(x)| > \lambda\}$.

By DCT, $\int_{E_\lambda} |f|\,dm \to 0$ as $\lambda \to \infty$.

$\Rightarrow \lambda d_f(\lambda) \to 0$ as $\lambda \to \infty$.

⑦

(A) True.

$$F(x) = \int_x^{x+1} g(y)\,dy$$

$$= (\chi_{[-1,0]} * g)(x).$$

$\therefore F$ is the convolution of two L^1 functions.

$\Rightarrow F \in L^1(\mathbb{R})$.

(B) True.

$$|F(x)| \leq \int_x^{x+1} |g(y)|\,dy \leq \|g\|_1 < \infty, \quad \forall x \in \mathbb{R}.$$

$\Rightarrow F$ is bounded.

(C) True.

$$F(x) = \int_{-\infty}^{x+1} g(y)\,dx - \int_{-\infty}^{x} g(y)\,dy.$$

$$\lim_{x\to\infty} F(x) = \int_{\mathbb{R}} g(y)\,dy - \int_{\mathbb{R}} g(y)\,dy = 0$$

& $\lim_{x\to-\infty} F(x) = 0 - 0 = 0$

$\Rightarrow \lim_{|x|\to\infty} F(x) = 0.$

⑧ (A) True

$$\int_0^1 |F(x)|\,dx \le \int_0^1 \left(\sum_{k=-\infty}^{\infty} |f(x+k)| \right) dx = \sum_{k=-\infty}^{\infty} \int_0^1 |f(x+k)|\, dx = \int_{\mathbb{R}} |f|\, dx < \infty .$$

(B) Not true, By (A).

C) Since $F \in L^1[0,1]$. F is finite a.e

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⑨ (A) Not true

(B) True

(C) Let $E_{k,\varepsilon} = \{ x : |f_k(x) - f(x)| > \varepsilon \}$.

$$\int_{\mathbb{R}} |f_k - f|\, dm \ge \int_{E_{k,\varepsilon}} |f_k - f|\, dm \ge \varepsilon \cdot m(E_{k,\varepsilon})$$

$$\Rightarrow m(E_{k,\varepsilon}) \le \frac{1}{\varepsilon} \int_{\mathbb{R}} |f_k - f|\, dm \to 0 \quad \text{as } k \to \infty$$

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⑩ (A) True

$$\int_{\mathbb{R}^n} |(f * g)(x)|\, dx \le \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x-y)\, g(y)|\, dy\, dx$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x-y)|\, |g(y)|\, dx \cdot dy \qquad (\because \text{integrant is } +\text{ve})$$

$$= \int_{\mathbb{R}^n} \|f\|_1 \cdot |g(y)|\, dy$$

$$= \|f\|_1 \cdot \|g\|_1 < \infty .$$

$$\Rightarrow f * g \in L^1(\mathbb{R}^n).$$

(B) True.

$$|f * g(x)| \le \int_{\mathbb{R}^n} |f(x-y)|\, |g(y)|\, dy$$

$$\le c \cdot \|f\|_1 . \qquad \text{where } c = \sup_{y \in \mathbb{R}^n} |g(y)|$$

$\Rightarrow f * g$ is bounded.

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